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Inflation and Global Equivalence

Geoffrey Martin
Department of Mathematics
University of Toledo
Toledo, OH 43606
gmartin@math.utoledo.edu

Abstract

This article investigates an extension of General Relativity based upon a class of lifted metrics on the cotangent bundle of space-time. The dynamics of the theory is determined by a fixed section of the cotangent bundle, representing the momentum of a fluid flow, and Einstein's equations for the fluid applied to the induced space-time metric on the submanifold of the cotangent bundle defined by the image of the section. This construction is formally analogous to the extension of Galilean Relativity by Special Relativity, and is shown to reduce to General Relativity as the gravitational constant approaches zero. By examining the consequences of the model for homogeneous cosmologies, it is demonstrated that this construction globalizes the equivalence principle, in that, the perfect fluid model of Special Relativity is sufficient to predict both the inflationary and the current era.

1 Introduction

In the standard formulation of General Relativity (GR), it is axiomatic that frames are non-dynamical objects determined by the choice of observers. This feature of GR is a consequence of the clock hypothesis axiom, which postulates that the rate at which an ideal clock (one that is corrected for the mechanical effects of acceleration) operates depends only on its 4-velocity. However, the role that frames play in dynamical structure of GR is more complicated than suggested by their place in the axiomatic framework. This is because matter, as it is modeled in the theory, often distinguishes a frame which becomes a dynamical object. Examples of such frames are the 4-velocity streamlines in fluid models and stationary observers in static space-times. The dependence of the space-time metric on the distinguished frame of the matter flow suggests an extension of GR in which the frame dependence of the metric is incorporated into the axiomatic structure of the theory.

To see the possibility of such an extension, a frame of observers must be viewed as a geometric object in its own right. This is best accomplished if

a frame, rather than being viewed as a choice of coordinates on space-time, is represented in an abreviated form, as a closed 1-form field λ on space-time M. The data that λ carries concerning the associated field of observers is the nature of the observers' clocks, and the local decomposition of M into space-like submanifolds. Now a 1-form field can also be viewed as a section of the cotangent bundle T^*M of M. From this perspective the geometric object most easily associated with a frame is its image $\lambda(M)$.

To arrive at the proposed extension of GR, the frame of observers $\lambda(M)$ is thought of as an analogue of an observer's world-line in Galilean or special relativity. The fundamental problem in special or Galilean relativity is to fix a representation of time along a world-line. A similar question arises in extending GR which is to establish method for defining a space-time metric on $\lambda(M)$. The Galilean approach advocates an extrinsic procedure in which there is a fixed correspondence that takes $\lambda(M)$ to a model space where distances are measured. In Galilean relativity the fixed extrinsic structure is the universal time line $\mathbf{R}$. In the present case, M and the bundle map $\pi \colon T^*M \to M$ serves this purpose. On the other hand, relativistic approach postulates that distance measurement is intrinsic to the submanifold $\lambda(M)$ itself, and may depend on the position of the submanifold in the ambient space.

The Galilean approach can be seen to yield the standard structure of GR. It is encoded in the axiom which for emphasis Einstein states as *continuum spatii et temporis est absolutum* [1]. In this article we shall investigate whether the relativistic approach may be the more powerful. Following Einstein's dictum, if the relativistic approach is to be a viable alternative, it must be demonstrated to be consistent, in the appropriate limit reduce to prior tested theories, and have interesting consequences. Consistency is of course a very difficult question in physics. The second criteria of reduction to existing theoretical structure at least suggests a level of consistency similar to existing theories. In following we hope to demonstrate that an extension of GR using the relativistic approach to measuring space-time distances satisfies the last two of these criteria.

To show that the relativistic approach does in fact have interesting consequences, we shall examine the implications of the extended theory for homogeneous cosmological models. Here, because in the extended theory the metric is frame dependent, it is most natural to apply the equivalence principle in the matter frame. This leads to a third rather than second order equation for the cosmic scale factor; the solutions of which appear to have interesting global consequences. It will be shown that if the universe is assumed to evolve towards a dust state, then it must have originated from an inflationary state. In fact the model suggests that the alternation of standard and inflationary phases may be a cyclic process. Analysis of the model shows that it has the same consequences as current homogeneous inflation models but with the advantage that cosmic evolution from the inflationary phase to the present can be described as a single process. The mathematical structure presented here does not yet apply to the study of fluctuation in initial cosmic matter density, and so it can not be expected to account adequately for phenomena such as reheating and condensation.

2 Extended General Relativity

In this discussion a frame on a space-time M will be represented by the pair (λ, V), where λ is 1-form field on M, and V is vector field on M. A frame provides a decomposition of the tangent space to M at p that is given by $TM_p = \mathrm{span}(V_p) \bigoplus \ker(\lambda_p)$. This decomposition is interpreted as the infinitesimal decomposition of TM_p into space and time directions by the observer associated with the frame and located at p. Note that this decomposition is independent of the choice of scale for the vectors V and λ. The scale of V and λ has useful descriptive properties. The scale V is associated with the observer's choice of time coordinates and the scale of λ can be interpreted as a mass scale for particles.

Although the association of 1-forms with mass is a standard correspondence in GR that is usually established from momentum consideration, we provide here an argument for this relation based on Newton's Second Law. Suppose that W is another vector field on M, then the invariant velocity w of W in the frame (λ, V) is given by $w = (1/\lambda(W))W$. Note that w is invariant quantity in that it is independent of the choice of scale of W, and hence is a geometric objects associated with the flow lines of W. The velocity w, however, is dependent on the choice of scale of λ. Newton's Law in the given frame may be expressed as $m\dot{w} = F(w)$, where F is the force and $\dot{w}$ is the invariant derivative of w. The invariant derivative is given in terms of the Levi-Civita connection ∇ by the expression

$$\dot{w} = \frac{1}{\lambda(W)}(\nabla_W \frac{W}{\lambda(W)})^\perp.$$

Here $^\perp$ is the projection onto the complement of V.

Now suppose that the frame (λ, V) is replaced by $(k\lambda, V)$ where k is a positive constant. Let w' be the invariant velocity of W in the frame $(k\lambda, V)$. The above definitions imply that $m\dot{w'} = m(1/k^2)\dot{w} = (1/k^2)F(w) = (1/k)F(w') =$ or $km\dot{w'} = F(w')$. Hence in the frame $(k\lambda, V)$ the mass of W is $m' = km$. Hence λ can be imputed to have the same dimension as m

The equivalence principle guarantees that relativistic dynamics can be reduced to Newtonian dynamics infinitesimally along any flow line of V. However, it is only under very strong conditions that this correspondence can be extended locally. In the case where $\lambda = m\ell(V)$, $\ell(V)$ being the 1-form metrically dual to V, the Newtonian constructions gives an equivalent description of metric geometry only when V is a gradient Killing field [5]. This assumption implies that V is holonomically trivial or $R(\cdot,\cdot)V = 0$. Unfortunately, a self gravitating frame of massive observers cannot be holonomically trivial, and this fact leads to a global incompleteness in the description of the physics of such systems. Mathematically, this difficulty surfaces in the lack of a globally holonomic frame in which to apply the equivalence principle. This mathematical fact is mirrored in physical principle that gravity destroys states of local thermodynamic equilibrium by invoking the other fundamental forces and thus altering the local

equation of state. This is particularly true in cosmology where the evolution of fluid streamlines can only be determined by fitting equations of state appropriate to each of the local temperature regime found in the evolving universe. The fact that there is no universal law that describes cosmic evolution thus can be traced to the fact that we lack a universal description of the forces in nature. However, the fact that such a simple model as the perfect fluid model appears to describe the bulk of cosmological space-time, suggests there should exist an underlying principle from which a globally significant cosmological model can be derived.

Here we shall investigate the possibility that such a principle may be derived from a generalization of the equivalence principle that arises in an extension of GR in which the space-time metric is frame dependent. This extension of GR is obtained by a procedure that is analogous to the procedure used to obtain special relativity (SR) from Galilean relativity (GLR). Recall that the Galilean theory of time can be formulated by supposing that the space-time manifold M possesses a fibration $t\colon M \to \mathbf{R}$. This fibration is assumed to possess a connection represented by a vector field V on M which must necessarily have the property that if ϕ_s is the flow of V then $\phi_{s_1}(t^{-1}(s_2)) = t^{-1}(s_1+s_2)$. Thus the pair (dt, V) determines a synchronous field of observers on M with the special property $dt(V) = 1$. Let N be a model for the fiber of t. GLR supposes that N and $\mathbf{R}$ are separately endowed with the complete Riemannian metric tensors q and g, and that g is solely responsible for determining the time between two events $p, q \in M$. If δ_g is the topological metric on $\mathbf{R}$ determined by g, then according to GLR, the time between p and q is given by $\delta_g(t(p), t(q))$. In SR time does not have this well defined character but depends on the observer with respect to which these events are referenced. The metric that an observer uses is induced on the observer world line from a metric $\widetilde{g}$ on M that is obtained by summing g and q. This is done using the decomposition $TM_p = \mathrm{span}(V_p) \bigoplus \ker(dt_p)$, the natural isomorphisms $t_*\colon \mathrm{span}(V) \to T\mathbf{R}$, and the identification $i_s\colon t^{-1}(s) \to N$. The metric $\widetilde{g}$ is then given by the expression $\widetilde{g} = -t^*g \bigoplus (1/c)^2 i^*q$ where c is a universal constant. In the limit as $c \to \infty$, SR reduces to GLR.

This procedure can be carried out again with the base manifold $\mathbf{R}$ replaced by the space-time manifold M, and the fibration $t\colon M \to \mathbf{R}$ replaced by a fibration $\pi\colon \overline{M} \to M$. The manifold $\overline{M}$ that replaces M can be discovered by noting that in SR the fiber $t^{-1}(s)$ parameterizes the positions of all possible observers at time s. Thus, by analogy it is natural to suppose that for an event $p \in M$ the fiber $\pi^{-1}(p)$ parameterizes the frames of all possible observers at p. The above description of frames indicates that the fiber should be $TM_p \times T^*M_p$. However, it will be assumed that there is a relation $\ell\colon TM \to T^*M$, and that the frames under consideration all have the form $(\lambda, \ell^{-1}(\lambda))$, and thus the fiber of π at p may be assumed to be T^*M_p, making $\overline{M} = T^*M$.

As in SR, a metric can be constructed on T^*M using the Lorentzian metric on M and the induced metric on the fibers of T^*M. To interpret this construction it is helpful to recall how relativity unifies space and time dimensions. In the case of SR, tangent vectors to N are assigned the dimension of length L

while tangent vectors to $\mathbf{R}$ are assigned the dimension of time T. Metric tensors are supposed to be dimensionless. If the universal constant c is supposed to have the dimension L/T, then for vector fields V and W on M, $\widetilde{g}(V, W)$ has dimension T^2. This suggests that the dimensional structure of M should be simplified by measuring the length of tangent vectors to N in T-units with 1 L-unit equaling $1/c$ T-units. In such a system $c = 1$ and all physical tangent vectors to M have dimension T.

In the case of T^*M, the above discussion of Newton's Law suggests that the fibers of T^*M should be given the dimension mass M. Thus in this case we see that there is the need of a physical constant $1/G$ of dimension M/T relating the base and fiber dimensions. The constant G is easily seen to have the same dimension as the gravitational constant.

To construct a metric on T^*M denote the space-time metric by g and proceed analogously to construct a metric $\widetilde{g}$ on T^*M. In analogy with GLR, the horizontal distribution H of the Levi-Civita connection plays the role of the field V. It is a well known fact in connection theory that TT^*M decomposes as $TT^*M_p = H_p \bigoplus VT^*M_p$, where VT^*M is the vertical distribution of the fibration $\pi\colon T^*M \to M$. Because T^*M is a vector bundle there is a natural isomorphism $i_p\colon V_p \to T^*M_{\pi(p)}$ defined for each $p \in T^*M$. The isomorphism i_p together with the isomorphism $\pi_{*p}\colon H_s \to TM_{\pi(p)}$ can be used to define a class of metrics on TT^*M with the property that H and VT^*M are orthogonal distributions. Of particular interest is the metric $\widetilde{g}$ given by

$$\widetilde{g}_p = -\frac{\epsilon}{g(p,p)}(i_p)^*g \bigoplus \frac{g(p,p)}{\epsilon}(\pi_{*p})^*g.$$

This expression is considerably more complicated than the corresponding expression in SR due to the presence of the metric dependent conformal factors. There are several reasons why this particular form was chosen for $\widetilde{g}$. First, the parallelism of this metric can be shown to be closely related to Fermi parallelism along curves on the space-time [4]. Thus, the relativistic effects of acceleration are encoded in the geometric structure defined by $\widetilde{g}$ [3]. Second, since $\widetilde{g}(X, Y)$ should have a well defined dimension, the dimensions of the vertical and horizontal summand of this quantity should be equal. This implies that $\epsilon/g(p,p)$ has dimension of the gravitational constant T/M, and so ϵ has the dimension of Plank's constant MT. Thus the dimension of the conformal factors can be factored as MT/M^2. This factorization is similar to the factorization of the gravitational constant $G = \hbar/{\mathsf{M}_P}^2$ where M_P is Planck's mass. This particular way of expressing the dimensional dependence of $\widetilde{g}$ will prove very useful in cosmological applications.

To interpret the metric $\widetilde{g}$, consider a field of observers associated with a section λ of T^*M. Just as in SR where the world-line of an observer inherits a metric that measures time displacement, in this case the image of space-time $\lambda(M)$ inherits a metric $G\widetilde{g}|_{\lambda(M)}$ which measures the space-time displacement in that frame. Here a dimensional constant G is introduced with dimension T/M so that when evaluated $G\widetilde{g}$ has dimension T^2. Thus in extended general relativity (EGR), an observer field λ detects not the space-time metric π^*g but

rather the space-time metric $G\widetilde{g}|_{\lambda(M)}$. It will be demonstrated in section 3 that if $G \to 0$ and $\epsilon \to 0$ in such a way that $Gm^2/\epsilon \to 1$, then EGR reduces to GR.

3 Global Equivalence in Friedmann Geometry

This section shall examine some consequences of EGR in the standard cosmological model. In this case, space-time M is modeled by the product $\mathbf{R} \times N$ where N is now a simply connected 3-dimensional space-form of constant sectional curvature κ. If q is the constant curvature metric on N and if t is the natural affine coordinate on $\mathbf{R}$, then consider the class of Friedmann metric tensors

$$g = -dt^2 \bigoplus S(t)^2 q,$$

where $S(t)$ is assumed to be a smooth positive function. Freidmann geometries possess a distinguished frame determined by the unit vector field T in the distribution $T\mathbf{R} \times 0$. This frame is described by the pair (T, s) where $s = m\ell(T)$ and m is some intrinsic mass. To study the geometry of the frame (T, s) first observe that $\widetilde{g}|_{s(M)}$ is of Robinson-Walker type. To determine the form of $\widetilde{g}$ on $s(M)$ use the induced map $s_*\colon TM \to Ts(M)$ to parameterize $Ts(M)$. It follows from the theory of linear connections that for any tangent vector $u \in TM_p$,

$$s_* u = \widetilde{u}_{s(p)} + {i_{s(p)}}^{-1} \nabla_u s,$$

where ∇ is the dual Levi-Civita connection of g and $\widetilde{\ }$ denotes the horizontal lift to TT^*M determined by ∇. From the identities

$$\begin{aligned} \nabla_T s &= 0 \\ \nabla_X s &= m \log \dot{S}(t) \ell(X), \end{aligned}$$

where X is a vector field in the distribution $0 \times TN$, and the above expressions for $s_* u$ and $\widetilde{g}$, it is easy to see that

$$\begin{aligned} \widetilde{g}(s_*T, s_*T) &= \frac{m^2}{\epsilon} \\ \widetilde{g}(s_*X, s_*Y) &= -\frac{m^2}{\epsilon}(1 - \frac{\epsilon^2}{m^2}(\log \dot{S}(t))^2) S(t)^2 q(X, Y). \end{aligned}$$

These expressions imply that since $\pi\colon s(M) \to \mathbf{R} \times N$ is a diffeomorphism

$$\widetilde{g}|_{s(M)} = \frac{m^2}{\epsilon}(\pi^* dt^2 \bigoplus -(1 - \frac{\epsilon^2}{m^2}(\log \dot{S}(t))^2) S(t)^2 \pi^* q).$$

Since the intrinsic geometry of the frame determined by s is a Friedmann geometry, it provides a model of a self gravitating perfect fluid. To complete the model one needs in addition to Einstein's equations an equation of state for the fluid. At this point it is customary in GR to appeal to the equivalence principle and to SR for an equation of state. In EGR, since the metric is frame

dependent, there is the question of choosing the frame in which to apply the equivalence principle. The hypothesis that shall be advanced is that the equivalence principle should be applied in the frame of the matter flow, and that in this frame it yields an equation of state with global validity. In other words, the entropy generating effects of gravity can be eliminated by "viewing" space-time from the matter frame. This idea can be thought of as an extension of Einstein's original version of the equivalence principle in which gravitational forces were eliminated by accelerated frames. The original form of the equivalence principle was too strong as it only applies infinitesimally or to pseudo-gravitational forces as note above. However, because the metric in EGR is frame dependent the original form of the equivalence is no longer limited to flat geometries and appears to have interesting implications.

To derive the consequences of this assumption recall that for a simple gas, kinematic analysis implies that the energy e, pressure p and density ρ satisfy the equations of state $e - \rho = \alpha p$ and that the energy momentum tensor has the form $T = (e+p)V \otimes V + pg$. When coupled with Einstein's equations, the equation of state yields a system of equations that determine $S(t)$. For simplicity let $S'(t)^2 = (1 - (\epsilon^2/m^2)(\log \dot{S}(t))^2)S(t)^2$. From Einstein's equations in the frame determined by s, we have

$$\begin{aligned} e &= \frac{3}{G}((\log \dot{S}'(t))^2 + \frac{\kappa}{S'(t)^2}) \\ p &= -\frac{1}{G}(2\log \ddot{S}'(t) + 3(\log \dot{S}'(t))^2 + \frac{\kappa}{S'(t)^2}). \end{aligned}$$

Further the matter conservation law implies that there is a constant g such that $\rho = g/S'(t)^3$.

To obtain the equations for $S(t)$ implied by these relations and the equation of state, first assume that, as suggested in the last section, $G = \epsilon/m^2$. It will also be useful to transform to the universal time scale by setting $u(t) = S(kt)$ where $k = \epsilon/m$. With these assumptions a calculation shows that $u(t)$ satisfies

$$\begin{aligned} &\log \dot{u}(1 - \log \dot{u}^2) lo\dddot{g}u = \\ &\quad \log \ddot{u}(1 - \log \dot{u}^2)(1 - \frac{3+4\alpha}{\alpha}\log \dot{u}^2) - \log \ddot{u}^2(1 - \frac{3+\alpha}{2\alpha}\log \dot{u}^2) + \\ &\quad \frac{3(1+\alpha)}{2\alpha}\log \dot{u}^2(1 - \log \dot{u}^2)^2 + \frac{3+\alpha}{2\alpha}\frac{\kappa k^2}{u^2}(1 - \log \dot{u}^2) + \frac{1}{2\alpha}\frac{k^2 G g}{u^3}(1 - \log \dot{u}^2)^{\frac{1}{2}}. \end{aligned}$$

It is physically reasonable to assume that the constants $(3 + \alpha/2\alpha)\kappa k^2$ and $(1/2\alpha)k^2 G g$ are both extremely small, and so the last two terms in the above expression can be neglected. If these terms are dropped then the above equation for u can be rewritten as a planar system in the variables $x(t) = \log \dot{u}(t)$ and $y(t) = \log \ddot{u}(t)$ which after rescaling time has the form $\dot{x} = F_x(x,y)$ and $\dot{y} = F_y(x,y)$ where

$$\begin{aligned} F_x &= x(1-x^2)y \\ F_y &= y(1-x^2)(1 - \frac{3+4\alpha}{\alpha}x^2) - y^2(1 - \frac{3+\alpha}{2\alpha}x^2) + \frac{3(1+\alpha)}{2\alpha}x^2(1-x^2)^2. \end{aligned}$$

This system, although it can not be explicitly solved, can be analyzed using the qualitative theory of planar systems. To interpret this analysis it should be kept in mind that the xy-phase plane is related through Einstein's equations to the Hubble constant-pressure plane of the cosmological model. First note that the above system possesses four stationary points $(0,0)$, $(-1,0)$, $(1,0)$ and $(0,1)$. The stationary point $(0,1)$ is easily seen to be of hyperbolic type. However the analysis of this equation is complicated by the fact that $(0,0)$ is semi-degenerate and $(-1,0)$ and $(0,-1)$ are totally degenerate. The stationary point (0,0) will be shown to be topologically of hyperbolic type and the stationary point at $(0,1)$ will be shown to possess elliptic and parabolic sectors. Using the symmetry of the equation it may be assumed that $x \geq 0$. This is equivalent to assuming that the Hubble constant is positive.

To understand the global behavior of solutions to this system first consider the structure of solutions near $(0,0)$. Since $F_x(0,y) = 0$, it follows that $x = 0$ defines a solution curve that adheres to $(0,0)$. To find the other adherent solution invoke polar coordinates by the standard relations $\dot{r} = F_r(r,\theta)$ and $r\dot{\theta} = F_\theta(r,\theta)$ where $F_r(x,y) = (xF_x(x,y) + yF_y(x,y))/r$ and $F_\theta(x,y) = (xF_y(x,y) - yF_x(x,y))/r$. The other adherent solution can be located by considering a sector bounded by $y = 0$ and a solution of $F_r(x,y) = 0$. Note that the equation $F_r(x,y) = 0$ is quadratic in y and may be solved to give radial isoclines

$$y_\pm = \frac{(1-x^2)(1-\frac{3+4\alpha}{\alpha}x^2) \pm \sqrt{(1-x^2)x^4(\frac{3+7\alpha}{\alpha} - 12\frac{1+\alpha}{\alpha}x^2) + (1-x^2)^2(1-x^2)^2}}{2(1-\frac{3+\alpha}{2\alpha}x^2)}.$$

We may distinguish radial isoclines by the property that $y_+(0) = 1$ and $y_-(0) = 0$. Let I_0 be the graph of y_- for $x \geq 0$. Consider a small sector W bounded by $y = 0$ and I_0. The vector field F exits both radial boundaries of W. First, because $F_x(x,0) = 0$ and $F_y(x,0) \geq 0$, F exits $y = 0$. Second, a Taylor expansion for $F_\theta(x,y)$ developed along I_0 implies that $F_\theta(x, y_-(x)) = -x^3 + o(x^4)$, and consequently, if the sector W is chosen sufficiently small, $F_\theta \leq 0$ on the segment of I_0 bounding W. As a result, there must either exist a unique solution or an interval of solutions that tend to $(0,0)$ as $t \to \infty$ with initial conditions on the arc bounding W.

To see that there is a unique solution, examine the first order ode $r(d\theta/dr) = \Psi(x,y) \equiv F_\theta(x,y)/F_r(x,y)$. Introduce the angular operator $\Theta = (x\partial/\partial y - y\partial/\partial x)/r$. It follows from a generalization of a lemma of Lonn that there is a unique solution adhering to $(0,0)$, if $\Theta\Psi < 0$ in the interior of W [6]. To verify this inequality, note that $\Theta\Psi(x,y) = Q(x,y)/F_r(x,y)^2$, where $Q(x,y)$ is a polynomial of order 14 in x and y with properties that $Q(x,y) = -4\alpha y^2(x^2 + y^2) + o(r^4)$ and $Q(x,0) = -3(5\alpha+3)(\alpha+3)x^6 + o(x^7)$. These facts imply that $Q(x,y)$ is negative definite in a neighborhood of the origin, which implies that if W is chosen small enough, then $\Theta\Psi < 0$ on the interior of W. Hence, there is a unique solution that adheres to $(0,0)$ in W, and determines a stable manifold J_0 for the stationary point $(0,0)$.

To understand the global backward time evolution of J_0, first consider the isoclines $F_y(x,y) = 0$. This equation is again quadratic in y and has the solu-

tions

$$\tilde{y}_{\pm} = (1-x^2)\left(\frac{(1-\frac{3+4\alpha}{\alpha}x^2) \pm \sqrt{12\frac{\alpha+1}{\alpha}x^4+(1-x^2)^2}}{2(1-\frac{3+\alpha}{2\alpha}x^2)}\right).$$

These isoclines again can be distinguished by $\tilde{y}_-(0)=0$ and $\tilde{y}_+(0)=1$. The most crucial feature of the graphs of $\tilde{y}_-$ and $\tilde{y}_+$ is the position and sign of the vertical asymptotes. It can be seen that only $\tilde{y}_-$ possesses an asymptote at $x_0 = \sqrt{2\alpha/(3+\alpha)}$ while $\tilde{y}_+$ is finite at x_0 and has the value $\tilde{y}_+(x_0) = (3(3-\alpha)(1+\alpha))/(4(3+\alpha)(7\alpha+3))$. The sign of the asymptote depends on whether $x_0>1$ or $x_0<1$. It is easily seen that $\lim_{x\to x_0^-} y_-(x) = -\infty$ if $x_0<1$ and $\lim_{x\to x_0^-} y_-(x) = \infty$ if $x_0>1$. Also note that both isoclines have the property that $y_{\pm}(\pm 1)=0$.

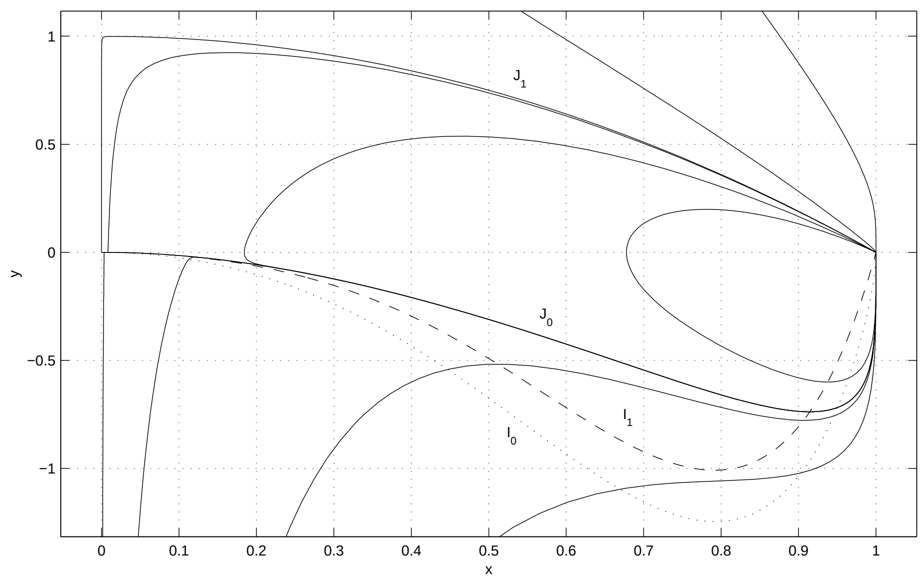

Figure 1. Flow lines for the equation $\dot{x} = yx(1-x^2)$, $\dot{y} = y(1-x^2)(1-4x^2) - y^2(1-(1/2)x^2) + (3/2)x^2(1-x^2)^2$ showing stable manifolds J_0 and J_1 and radial and cartesian isoclines I_0 and I_1.

To understand the relation between the stable manifold J_0 and the isocline I_1 determined by the graph of $\tilde{y}_-$, we develop a Taylor expansion for J_0 at $(0,0)$. We may assume that J_0 can be parameterized as the graph of a smooth function $f(x)$; where f must satisfy $F_y(x,f(x)) = f'(x)F_x(x,f(x))$. To obtain the first two terms in the Taylor expansion, assume that $f(x) = Cx^2 + Dx^4 + o(x^5)$ and substitute into the above expression. It can be seen that $f(x) = -(3(1+\alpha))/(2\alpha)x^2 + (3(1+\alpha))^2/(2\alpha)^2x^4 + o(x^5)$ while a Taylor expansion for $\tilde{y}_-$ at

$(0,0)$ has the form $\tilde{y}_-(x) = -(3(1+\alpha))/(2\alpha)x^2 - (3(1+\alpha))^2/(2\alpha)^2x^4 + o(x^5)$. Thus the stable manifold J_0 lies above isocline I_1.

Consequently, the global behavior of J_0 is determined by the position of the asymptote x_0. If $\alpha < 3$ or $x_0 < 1$, J_0 lies in an unbounded region defined by $y = 0$, $x = 1$ and I_0 in which case $F_y > 0$ and $F_x > 0$. Since $x = 1$ is a solution curve J_0 cannot intersect it , and thus is asymptotic to it as $t \to -\infty$. The case where $\alpha > 3$ or $x_0 > 1$ is quite different. Here J_0 lies in a bounded region U defined by $y = 0$ and I_1 (see Fig. 1). In this case divide I_1 into two segments I_1' and I_1'' at the point (x_1, y_1) where I_1 attains its minimum y-value. Assume $(0,0) \in I_1'$. It is easy to see that the vector field F exits U along $y = 0$ and I_1' and enters along I_1''. Consequently, in backward time J_0 either adheres to the stationary point $(1,0)$ or leaves U along I_1''. In latter case since $F_y(x, y_1) < 0$ for $x_1 < x < 1$, and since $x = 1$ is a solution, it follows that, if J_0 enters the region bounded by I_1'', $x = 1$ and $y = y_1$, it must adhere to $(1,0)$. Thus in the case where $\alpha > 3$ the manifold J_0 is an unstable manifold for $(1,0)$ and a stable manifold for $(0,0)$.

The unstable manifold J_1 leaving the stationary point $(0,1)$ can also be seen to adhere to (1,0). This follows from the fact that the boundary of the unit square $[0,1] \times [0,1]$ either consists of solutions curves, $x = 0$ and $x = 1$, or segments where the vector field is entering. Since J_1 enters the unit square, it must adhere to a singular point on the boundary. However, it cannot adhere to $(0,0)$, since the only inward pointing characteristic direction with angle between 0 and $\pi/2$ is uniquely associated with J_0. Thus J_1 must adhere to $(1,0)$.

The final step in understanding the global behavior of the flow of F is to analyze the stationary point $(1,0)$. Here we shall restrict our attention to the case where $\alpha \to \infty$; that is to zero pressure cosmologies. In this limit we find that

$$\begin{aligned} F_x &= x(1-x^2)y \\ F_y &= y(1-x^2)(1-4x^2) - y^2(1-\frac{1}{2}x^2) + \frac{3}{2}x^2(1-x^2)^2. \end{aligned}$$

To study the stationary point at $(1,0)$ introduce the coordinate change $z = 1 - x$ and transform to polar coordinates letting $z = r\cos(\theta)$ and $y = r\sin(\theta)$. After the standard rescaling of time, one obtains the equation for the blow-up of the vector F at $(1,0)$; namely $\dot{r} = R(r,\theta)$ and $\dot{\theta} = S(r,\theta)$ where $R(r,\theta) = rR_0(\theta) + \eta(r,\theta)$ and $S(r,\theta) = S_0(\theta) + \xi(r,\theta)$ with $\lim_{r\to 0}\xi(r,\theta) = 0$ and $\lim_{r\to 0}\eta(r,\theta)/r = 0$. The characteristic directions are determined by the equation $S_0(\theta) = 0$. The solutions to this equation are easily obtained since

$$\begin{aligned} S_0(\theta) &= \frac{15}{4}\cos(\theta)(1+\cos(2\theta+\varphi)) \\ R_0(\theta) &= \frac{15}{4}\sin(\theta)(\frac{7}{15}+\cos(2\theta+\varphi)) \end{aligned}$$

where $\varphi = \arctan(4/3)$. The equation $S_0(\theta) = 0$ is easily seen to have solutions $\pi/2$, $-\pi/2$, $(\pi-\varphi)/2$ and $(\pi-\varphi)/2+\pi$. Also a short calculation gives $(\pi-\varphi)/2 = \arctan(2)$.

To determine whether there are solutions to the ode adhering to $(1,0)$ along these characteristic directions, first note that at any of these angles $R(\theta) \neq 0$. Consequently, the existence of such solutions is determined by the Taylor expansion of S_0 at the given characteristic angle. In general, if the first non-vanishing coefficient is of odd degree then there exist solutions to the ode adhering along that characteristic direction. However, if the first non-vanishing coefficient is even then there exist either no solutions or an interval of solutions adhering along that characteristic direction. Examination of the above expression for S_0 shows that the Taylor expansions of S_0 at $\pi/2$ and $-\pi/2$ are of odd degree, and at $\arctan(2)$ and $\arctan(2)+\pi$ they are of even degree. In both cases there are solutions adhering to $(1,0)$ along these characteristic directions [2].

Any solution with initial conditions $(x,0)$ for $0 \leq x \leq 1$ must adhere in forward time to $(1,0)$ along the characteristic direction $\arctan(2)$ or $\pi/2$, and in backward time along the characteristic direction $-\pi/2$. Since solutions adhere to $\arctan(2)$ exist, there must exist a closed interval $[x_0,1]$ such that if $x \in [x_0,1]$ then solutions with initial conditions $(x,0)$ adhere to $(1,0)$ along the $\arctan(2)$ direction. Numerical results suggest that in fact $x_1 = 0$, and that the last solution adhering to $(1,0)$ at the angle $\arctan(2)$ is J_1. Hence it is conjectured that the solution curves $x=0$ for $0 \leq y \leq 1$, J_0 and J_1 form a separatrix for an elliptic sector of the stationary point $(1,0)$.

To develop a physical interpretation for these results, first note that the rescaling $u(t) = S(kt)$ and Einstein's equations imply that points in xy-plane are related to the energy and pressure of cosmological states by the relations

$$\begin{aligned} e &= \frac{m^4}{\epsilon^2} 3x^2 \\ p &= -\frac{m^4}{\epsilon^2}(2y+3x^2). \end{aligned}$$

The most important integral curve of the vector field F for the interpretation of these results is J_0. It not only approximates the standard cosmological model for large time, but also, in the case where $\alpha \geq 3$, it originates from an inflationary state. To see this, rescale vector field F so that its parameter correctly reflects the time evolution of the original problem; that is $\dot{x} = y$. This rescaling can be accomplished by representing the solution curve as the graph of $f(x)$, and then solving for $x(t)$ by integrating $\dot{x} = f(x)$. In the case of J_0 at $(0,0)$, from above, we know that f has the form $f(x) = -((3+3\alpha)/2\alpha)x^2 + g(x)$ where $\lim_{x\to 0} g(x)/x^3 = 0$. Consequently, integration gives that $x(t) = (2\alpha/(3+3\alpha))(1/t) + b(t)$ where $\lim_{t\to\infty} b(t) = 0$. But, this is just the asymptotic behavior of solutions to the standard model with sectional curvature $\kappa = 0$ and density $\rho \ll 1$. Thus J_0 or solutions close to J_0 are candidates for equations of state that give reasonable cosmologies. Following J_0 in backward time, it either originates from a state of infinite pressure in the case where $\alpha < 3$, or a state where $p = -e$ in the case where $\alpha \geq 3$. The latter equation of state is the equation for inflationary cosmology. The fact that for $\alpha > 3$, cosmological models describe cold matter weakly interacting with radiation indicates that, in this scenario, the inflationary phase is predicted by the current condition of the universe. Note

that the energy density at the singularity is $3m^4/\epsilon^2$. If m is taken to be the Planck mass this is consistent with the chaotic inflation hypothesis.

To gain a better insight into the significance of this model for inflationary cosmology, requires a more detailed examination of the solutions near the stationary point $(1,0)$. Using a similar analysis as was applied at $(0,0)$, it can be shown that solution curves that adhere to $(1,0)$ are the graphs of either $f(z) = -z^{1/4}C(z)$ where C is analytic and $C(0) > 0$ or $g(z) = 2z + h(z)$ where $\lim_{z\to 0} h(z)/z = 0$. These solutions correspond to the $-\pi/2$ and $\arctan(2)$ characteristic directions respectively. To find $x(t)$ solve $\dot{z} = -f(z)$ and set $x = 1-z$. In first case with the initial condition (t_0, z_0), integration gives for $0 < z < z_0$, $z^{3/4}\widetilde{C}(z) - z_0^{3/4}\widetilde{C}(z_0) = t - t_0$ where $\widetilde{C}(0) = 4/(3C(0))$. Let $u^{4/3}D(u^{4/3})$ be the inverse of $z^{3/4}\widetilde{C}(z)$ with $D(0) = 1/\widetilde{C}(0)^{4/3}$, then the solution $x(t)$ has the form

$$x(t) = 1 - ((t-t_0) + z_0^{\frac{3}{4}}\widetilde{C}(z_0))^{\frac{4}{3}} D(((t-t_0) + z_0^{\frac{3}{4}}\widetilde{C}(z_0))^{4/3}).$$

One significant observation that follows from this expression is that, if J_0 is parameterized by the natural time parameter, then the stationary point is reached along J_0 in finite time. In fact, since $z_0^{\frac{3}{4}}\widetilde{C}(z_0)) > 0$ the singularity is reached at time $t_1 < t_0$, $t_1 = t_0 - z_0^{3/4}\widetilde{C}(z_0)$. Knowing $x(t)$ we may also calculate the scale factor $u(t) = \dot{x}(t)/x(t)$. Integration gives

$$u(t) = u(t_1)\exp((t-t_1) - ((t-t_0) + z_0^{\frac{3}{4}}\widetilde{C}(z_0))^{\frac{7}{3}}\widetilde{D}(((t-t_0) + z_0^{\frac{3}{4}}\widetilde{C}(z_0))^{4/3}).$$

The analysis of this expression shows that, although for small $t - t_1$, $u(t)$ grows exponentially, exponential growth does not persist long enough to account for inflation. To find true inflationary growth in this model we examine the behavior of solutions adhering to $(1,0)$ along the characteristic direction $\arctan(2)$.

For solutions adhering along the $\arctan(2)$ direction, the integral of $\dot{z} = -g(z)$ can be expressed in terms of the initial data (t_0, z_0) and a bounded function $b(z)$, and has the form $zb(z) = z_0 b(z_0)e^{-2(t-t_0)}$ for $0 < z < z_0$. If the inverse of $zb(z)$ is $vd(v)$, where d is also bounded, then $x(t)$ has the form

$$x(t) = 1 - z_0 b(z_0)e^{-2(t-t_0)}d(z_0 b(z_0)e^{-2(t-t_0)}).$$

Hence the scale factor $u(t)$ has the form

$$u(t) = \exp((t-t_0) + z_0 b(z_0)e^{-2(t-t_0)}\tilde{d}(z_0 b(z_0)e^{-2(t-t_0)})).$$

From this expression we see that as $t \to \infty$ the scale factor undergoes exponential growth. This fact suggests that the solutions of the elliptic sector may be given physical interpretation if it is supposed that the inflationary phase of the universe's development was the final state of a previous cycle.

As final remarks, observe that both the flatness of space and the adherence to the critical density can be deduced from this model, but in a manner quite different from the way in which these conclusions arise in standard inflationary models. First, the fact that the density of matter must be extremely close to

critical density is a consequence of the age of the universe relative to universal time scale k. Since, if cosmic evolution did not follow a path extremely close to J_0, it would have suffered a catastrophe before its current age. The flatness of space is a consequence of the same scaling law, because for large time scales the curvature may be neglected as it is multiplied by a factor of k^2. Finally, observe that the stationary point at $(1, 0)$ corresponds to singularity in $\widetilde{g}|_{s(M)}$. Notice that if $x^2 = 1$, then $S'(t) = 0$. At such points $Ts(M)$ is tangent to the null cone of $\widetilde{g}$ and so space-time at the singularity does not have a well defined spatial metric.

4 Submanifold geometry for a geodesic observer field

In the previous section the implications of EGR are studied for the simplest interesting example, namely Friedmann geometries. In general it is difficult to compute the intrinsic geometry of an arbitrary submanifold $s(M)$ of T^*M. Although general formulas for the curvature of such submanifolds can be developed, they are not at this point particularly illuminating and so will not be fully exposed here. Rather, this section will demonstrate that if s has constant length and $\ell^{-1}(s)$ is a geodesic vector field, then the difference between the curvature on $s(N)$ induced by π^*g and $\widetilde{g}$ depends only on the first three covariant derivatives of s, and can be developed in a power series in ϵ with first non-vanishing term of order ϵ^2. Thus, the additional terms that appear in the Einstein tensor when $\widetilde{g}$ is used in the place of π^*g can be viewed as additions to energy momentum tensor that are coupled to gravity by higher powers of the gravitational constant.

To understand the intrinsic curvature of $s(M)$, first calculate the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ on T^*M. Let R be the curvature tensor of the dual Levi-Civita connection in T^*M. By metric duality R determines a tensor $\bar{R}$ defined on $T^*M \times T^*M \times TM$ with values in TM by the expression $g(\bar{R}(s,t)v, w) = g(R(v,w)s, t)$. For vector fields X and Y and 1-form fields s and t, $\widetilde{\nabla}$ is determined at a point $p \in T^*M$ by the expressions

$$
\begin{aligned}
\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}_p &= \widetilde{\nabla_X Y} + \frac{1}{\epsilon} g(p,p) g(X,Y) i^{-1}p - i^{-1}\frac{1}{2} R(X,Y)p \\
\widetilde{\nabla}_{i^{-1}t}\widetilde{Y}_p &= \frac{g(y,p)}{g(p,p)}\widetilde{X} - \frac{\epsilon^2}{2g(p,p)^2}\bar{R}\widetilde{(p,t)}X \\
\widetilde{\nabla}_{\widetilde{X}} i^{-1}t_p &= i^{-1}\nabla_X t + \frac{g(t,p)}{g(p,p)}\widetilde{X} - \frac{\epsilon^2}{2g(p,p)^2}\bar{R}\widetilde{(p,t)}X \\
\widetilde{\nabla}_{i^{-1}t} i^{-1}s_p &= -\frac{1}{g(p,p)}(g(p,t)i^{-1}s + g(p,s)i^{-1}t - g(s,t)i^{-1}p).
\end{aligned}
$$

Now fix a section s of T^*M that has constant length and is closed. Since $\nabla.s$ is a bundle morphism between TM and T^*M, define $\nabla.s^T$ to be its transpose;

that is, $\nabla . s^T$ is a morphism between T^*M and TM. The conditions that s be of constant length and closed can be stated in the form $\nabla_s s^T = 0$ and $\nabla_U s = \ell \nabla_{\ell U} s^T$ where U is a vector field on M and $\ell \colon TM \to T^*M$ is the identification map determined by the metric g. Note that these identities immediately imply the classical result that $\ell^{-1}s$ is a geodesic vector field. If s is assumed to be time-like and $m^2 = -g(s,s)$ then for vector fields U and W on M the above expressions for $\widetilde{\nabla}$ imply that

$$\begin{aligned}
\widetilde{\nabla}_{s_*U} s_* W &= \widetilde{\nabla}_{\widetilde{U}+i^{-1}\nabla_U s} \widetilde{W} + i^{-1}\nabla_W s \\
&= \widetilde{\nabla_U W} + \frac{\epsilon^2}{2m^4}\Big(\bar{R}(s,\widetilde{\nabla_U})W + \bar{R}(s,\widetilde{\nabla_W})U\Big) - \frac{1}{2} i^{-1} R(U,W)s + \\
&\qquad i^{-1}\nabla_U \nabla_W s - \left(\frac{m^2}{\epsilon^2} g(U,W) + \frac{1}{m^2} g(\nabla_U s, \nabla_W s)\right) i^{-1}s.
\end{aligned}$$

After some calculation using this form for $\widetilde{\nabla}_{s_*U} s_* W$, one obtains the following expression for the sectional curvature $\widetilde{K}(s_*U, s_*W) = \widetilde{g}(\widetilde{R}(s_*U, s_*W)s_*U, s_*W)$ of $\widetilde{g}$ along $s(N)$

$$\begin{aligned}
&\widetilde{K}(s_*U, s_*W) = \\
&\quad -\frac{m^2}{\epsilon} K(U,W) + \frac{\epsilon}{m^2}\Big(g(\nabla_U R(U,W)s, \nabla_W s) + \\
&\quad g(\nabla_W R(W,U)s, \nabla_U s) + g(R(U,W)\nabla_U s, \nabla_W s)\Big) + \\
&\quad \frac{3\epsilon}{4m^2} g(R(U,W)s, R(U,W)s) + \frac{\epsilon^3}{4m^4}\Big(g(\bar{R}(s,\nabla_U s)W, \bar{R}(s,\nabla_W s)U) + \\
&\quad g(\bar{R}(s,\nabla_U s)W, \bar{R}(s,\nabla_U s)W) + g(\bar{R}(s,\nabla_W s)U, \bar{R}(s,\nabla_W s)U) - \\
&\quad 3g(R(s,\nabla_U s)U, \bar{R}(s,\nabla_W s)W)\Big) + \frac{\epsilon}{m^4}\Big(g(\nabla_U s, \nabla_W s)^2 - \\
&\quad g(\nabla_U s, \nabla_U s) g(\nabla_W s, \nabla_W s)\Big) - \frac{m^4}{\epsilon^3}\Big(g(U,W)^2 - g(U,U)g(W,W)\Big).
\end{aligned}$$

To obtain the intrinsic sectional curvature of $s(M)$ from this expression requires the second fundamental form of $s(M)$, $\alpha(U,W) = P'^{\perp}(\widetilde{\nabla}_{s_*U} s_* W)$. The orthogonal projection $P'^{\perp}$ on to the compliment of $Ts(M)$ can be expressed in terms of covariant derivatives of s. To see this note that the tangent space to $s(N)$ and its compliment are described as the graphs of the endomorphisms $C \colon H_s \to VT^*M_s$ and $C^* \colon VT^*M_s \to H_s$; that is, $Ts(M)_s = H_s \oplus C(H_s)$ and $Ts(M)_s^{\perp} = VT^*M_s \oplus C(VT^*M_s)$. It is easy to see that for a vector field U, $C(\widetilde{U}) = i^{-1}\nabla_U s$ and $C^*(i^{-1}t) = (\epsilon^2/m^4)\widetilde{\nabla_t s^T}$. The projection $P'^{\perp}$ can be expressed in terms of C and C^*, and the orthogonal projections P and $P^{\perp}$ that map TT^*M onto H and its orthogonal compliment VT^*M, and has the form

$$P'^{\perp} = (P^{\perp} + C^*P^{\perp})(P^{\perp} - CC^*P^{\perp})^{\#}(P^{\perp} - CP)$$

where $^{\#}$ denotes the group inverse.

The assumption that s is of constant length and closed implies that $\ell\nabla.s$ has the structure of a 3-dimensional symmetric linear map A, and hence satisfies the polynomial identity $A^3 = \xi_1 A^2 - \xi_2 A + \xi_3$. The fundamental invariants ξ_1, ξ_2 and ξ_3 are given in terms of the traces of powers of A by the expressions

$$\begin{aligned}
\xi_1 &= \mathrm{tr}(A) \\
\xi_2 &= \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)) \\
\xi_3 &= \frac{1}{6}\mathrm{tr}(A)^3 - \frac{1}{2}\mathrm{tr}(A)\mathrm{tr}(A^2) + \frac{1}{3}\mathrm{tr}(A)^3.
\end{aligned}$$

These invariants may be used to reduce $(1-A^2)^{-1}$ to the expression

$$\begin{aligned}
(1-A^2)^{-1} = &\\
&\frac{1+\xi_1 - 2\xi_2 + \xi^2}{1+\xi_1^2 - 2\xi_2 + \xi_1\xi_3 + \frac{1}{3}\xi_1^3\xi_3 - \frac{2}{3}\xi_1\xi_2\xi_3 + \frac{1}{3}\xi_3^2} + \\
&\frac{\xi_3 - \xi_1\xi_2}{1+\xi_1^2 - 2\xi_2 + \xi_1\xi_3 + \frac{1}{3}\xi_1^3\xi_3 - \frac{2}{3}\xi_1\xi_2\xi_3 + \frac{1}{3}\xi_3^2}A + \\
&\frac{1+2\xi_1^2 + 3\xi_2}{1+\xi_1^2 - 2\xi_2 + \xi_1\xi_3 + \frac{1}{3}\xi_1^3\xi_3 - \frac{2}{3}\xi_1\xi_2\xi_3 + \frac{1}{3}\xi_3^2}A^2.
\end{aligned}$$

To use these identities to write an expression for ${P'}^{\perp}$ in closed form, introduce $\widetilde{\ell}\colon H \to VT^*M$ defined by $\widetilde{\ell}(\widetilde{U}) = i^{-1}\ell(U)$, and endomorphisms B^* and B of VT^*M and H given by $B^*(i^{-1}t) = i^{-1}\nabla_{\ell^{-1}t}s$ and $B(\widetilde{U}) = \widetilde{\ell^{-1}\nabla_U s}$. If A is taken to be either $(\epsilon/m^2)B^*$ or $(\epsilon/m^2)B$ then, since both maps have the same invariants, ${P'}^{\perp}$ can be seen to be of the form

$$\begin{aligned}
{P'}^{\perp} = &\\
&(1+p_1+p_2\frac{\epsilon}{m^2}B^* + p_3(\frac{\epsilon}{m^2}B^*)^2)P^{\perp} - \frac{m^2}{\epsilon}(q_1+q_2\frac{\epsilon}{m^2}B^* + q_3(\frac{\epsilon}{m^2}B^*)^2)\widetilde{\ell}P \\
&+\frac{\epsilon}{m^2}(q_1+q_2\frac{\epsilon}{m^2}B + q_3(\frac{\epsilon}{m^2}B)^2)\widetilde{\ell}^{-1}P^{\perp} - (p_1+p_2\frac{\epsilon}{m^2}B^* + p_3(\frac{\epsilon}{m^2}B^*)^2)P
\end{aligned}$$

where p_1, p_2, p_3, q_1, q_2, q_3 have explicit rational expressions in terms of the invariants and have the following orders in ϵ; $p_1 = O(\epsilon^4)$, $p_2 = O(\epsilon^3)$, $p_3 = 1+O(\epsilon^2)$, $q_1 = O(\epsilon^3)$, $q_2 = 1+O(\epsilon^2)$, $q_3 = O(\epsilon)$.

The Gauss equation states that if K_s is the intrinsic sectional curvature of $s(N)$, then K_s is related to the ambient sectional curvature by

$$\begin{aligned}
&K_s(s_*U, s_*W) = \\
&\quad\widetilde{K}(s_*U, s_*W) - \widetilde{g}(\alpha(s_*U, s_*W), \alpha(s_*U, s_*W)) + \widetilde{g}(\alpha(s_*U, s_*U), \alpha(s_*W, s_*W)).
\end{aligned}$$

To isolate the terms of order less than ϵ that are present in this expression first decompose $\nabla_{s_*U}s_*W$ as $\nabla_{s_*U}s_*W = T(s_*U, s_*W) + S(s_*U, s_*W)$ where T is vertical and S is horizontal. Since $\nabla_s s^T = 0$, it follows that $B^* i^{-1}s = 0$, and

so the only terms of order less than ϵ that contribute to K_s are of the following type

$$\begin{aligned}
&\widetilde{g}(\alpha(s_*U, s_*W), \alpha(s_*V, s_*Z)) = \\
&\qquad \widetilde{g}(T(s_*U, s_*W), T(s_*V, s_*Z) + \widetilde{g}(T(s_*U, s_*W), B^*\widetilde{\ell}S(s_*V, s_*Z)) + \\
&\qquad \widetilde{g}(B^*\widetilde{\ell}S(s_*U, s_*W), T(s_*V, s_*Z) + O(\epsilon) \\
&= \quad -\frac{m^4}{\epsilon^3} g(U, W) g(V, Z) + O(\epsilon).
\end{aligned}$$

Hence, the Gauss equation gives that $K_s(s_*U, s_*W) = -(m^2/\epsilon)K(U, W) + O(\epsilon)$. In fact to order ϵ, a calculation shows that

$$\begin{aligned}
&K_s(s_*U, S_*W) = \\
&\quad -\frac{m^2}{\epsilon} K(U, W) + \frac{\epsilon}{m^2}(g(R(U, W)s, R(U, W)s) - g(\nabla_W R(U, W)s, \nabla_U s) + \\
&\quad g(\nabla_U R(U, W)s, \nabla_W s) + 3g(R(U, W)\nabla_U s, \nabla_W s) + \\
&\quad g(\nabla_W(\nabla_U s), \nabla_U(\nabla_W s)) - g(\nabla_W(\nabla_W s), \nabla_U(\nabla_U s))) + o(\epsilon).
\end{aligned}$$

Thus the terms that must be added to $K(U, W)$ to obtain $K_s(s_*U, s_*W)$ occur at two order of ϵ higher than $K(U, W)$ and involve $\nabla.s$, $\nabla.(\nabla.s)$ and $\nabla.(\nabla.(\nabla.s))$ as $R(U, W)s = \nabla_U(\nabla_W s) - \nabla_W(\nabla_U s)$.

It is interesting to note that if GK_s is truncated at order ϵ^2, and is used to construct approximate solutions for Friedmann geometries as in Section 2, then the solution curve J_0 does not return to the infationary state but rather recedes to infinity in backward finite time for all values of α.

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